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A continuation and existence result for a boundary value problem on an unbounded domain arising for the electrical potential in a cylindrical double layer ☆

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Abstract

Using a continuation theorem for contractions, the existence, uniqueness, and an approximation method for a class of nonlinear boundary value problems on an unbounded interval is obtained. The results apply in particular to the problem in the title.

Keywords: Poisson–Boltzmann equation; Electric potential distribution; Contraction mapping principle; Continuation theorem; Leray–Schauder alternative; Modified Bessel functions; Weighted function space

1. Introduction

One of the features of Banach's fixed point theorem is that it not only gives the existence and uniqueness of solutions of many complicated problems but additionally provides a (theo-

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retical) algorithm to approximate the solutions. The drawback is that its assumptions are rather restrictive: One needs a contraction and, in addition, a closed invariant set under this contraction.

An example where the latter is too restrictive arises in chemical engineering problems, see e.g. [6–8], of a type which we describe now. These problems arise for instance in the study of the electric potential distribution for the case of an infinitely long cylindrical surface [7]. The mathematical modelling of this problem leads to the classical Poisson–Boltzmann equation [8]

$$\Delta_r u(r) = f\big(u(r)\big) \tag{1.1}$$

for the unknown electric potential u on an unbounded domain $[R, \infty)$ (with fixed known $R > 0$) where

$$\Delta_r u(r) = \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}(r)\right)$$

denotes the radial Laplace operator. The natural boundary values associated with the problem are

$$u(R) = u_0, \qquad \lim_{r\to\infty} u(r) = 0. \tag{1.2}$$

In the most important special case, the nonlinearity f has the form

$$f(u) := \alpha \sinh(au) \tag{1.3}$$

where $\alpha, a > 0$ are known constants. However, also other (smooth) nonlinearities f are of interest which all have in common that they satisfy $f(0) = 0$ and $f'(0) > 0$.

In the paper [1], problem (1.1)/(1.2) was studied by inverting a certain approximation of the linearization of problem (1.1) and applying Banach's fixed point theorem for the arising fixed point problem. One might consider this either as some simplified Newton's method or as some form of Schauder's linearization trick. However, although this approach provides a uniqueness result in a reasonable wide class of functions, the corresponding existence result and the convergence of the algorithm of successive approximation can only be proved for very small values of the boundary value $|u_0|$. The reason is the above mentioned difficulty: Although the operator obtained by this approach is a contraction on a natural set in an appropriate function space, it is very hard to find a set which is mapped into itself. Actually, the only canonical choice for such a set is a ball around a fixed point of the linearization. It turned out that this ball is very small which in [1] essentially leads to the mentioned requirement that $|u_0|$ must be small.

We are therefore interested in finding alternatives to the a priori knowledge of an invariant set and thus to obtain the existence of a solution also for larger values of $|u_0|$. To this end, we observe that the problem of finding invariant sets occurs also in topological results like Schauder's fixed point theorem. For these results this problem can often be circumvented by applying instead degree theory or at least some so-called continuation theorems. The famous Leray–Schauder alternative is the most popular special case of such a theorem. These results are of the type that one only has to verify that a certain homotopy is admissible, i.e. that it has no fixed points on the boundary of the domain.

However, we do not want to apply topological methods here, because we simultaneously want to have a constructive method, i.e. a (theoretical) method to approximate the solution. Moreover, the Leray–Schauder degree cannot be directly applied in our case because the obtained operator is not compact; even more advanced degree theories (like degree theories for condensing maps) cannot be applied straightforwardly in our case.

In Section 2, we prove a general continuation result for contractions and formulate the special case of the Leray–Schauder alternative which we will use, and we sketch the corresponding algorithm which can be used to obtain the fixed point. This Leray–Schauder alternative will be applied for problem (1.1)/(1.2) in Section 4. In order to do so, we recall some facts about the linearization of the problem in Section 3. Finally, in Section 5, we discuss the hypotheses of our result for the special case (1.3). Our main result will be that, under almost the same assumptions which ensure that the obtained operator is a contraction, problem (1.1)/(1.2) actually *has* a solution and that this solution can be approximated by a certain algorithm. Thus, roughly speaking, under almost the same hypothesis under which we could prove in [1] only the uniqueness of the solution, we obtain now also its existence. Of course, the algorithm to approximate the solution is now not only a straightforward successive approximation but slightly more complicated.

2. Continuation and Leray–Schauder alternative for contractions

It might appear superfluous to prove a Leray–Schauder alternative for contractions. Indeed, since each contraction is in particular a set-contraction for the Kuratowski measure α of noncompactness, i.e.

$$\alpha\big(F(M)\big) \leqslant q\alpha(M) \quad (M \subseteq \operatorname{dom} F) \tag{2.1}$$

with the contraction constant $q \in [0, 1)$, one might expect that each contraction (in a Banach space) is condensing, i.e.

$$\alpha\big(F(M)\big) < \alpha(M) \quad \text{if } \overline{F(M)} \text{ is not compact} \tag{2.2}$$

and so the Leray–Schauder alternative is a simple special case of the Nussbaum–Sadovskiĭ degree theory for condensing maps, see e.g. [10,11]. However, this argument is valid only on *bounded* subsets of a Banach space (i.e. if $\operatorname{dom} F$ or at least $\operatorname{rng} F$ is bounded): A map F on an unbounded subset of a Banach space (with unbounded range) is *never* condensing, since for $M = \operatorname{dom} F$ both sides of (2.1) are infinite which means that (2.2) fails. In such a case, there is no degree theory available for F. Nevertheless, if one knows a priori bounds for the fixed points, one may apply degree theory in a neighborhood of the fixed point set anyway and thus obtain the corresponding continuation results. This is even possible in a more general setting than degree theory [3].

However, a direct result for contractions has some other nice features. For one, the underlying metric space X may fail to satisfy "nice" topological properties (like being a locally finite union of convex sets, which is currently still the mildest requirement on X under which a degree theory for condensing maps is known [9]). Secondly, and more important to us, the proof is constructive and leads to an at least theoretical algorithm which we sketch at the end of the section. In fact, our proof and (2.4) might even be used to obtain estimates on how close the current approximation is to the exact solution. Finally, in contrast to the topological approaches, the following theorem does not require a priori compactness of the fixed point set but only an a priori equicontinuity which might be much easier to verify for certain homotopies (although this is not the case for our Leray–Schauder alternative below).

A result similar to the following continuation theorem was obtained in [5]. However, in [5] a global Lipschitz condition with respect to the homotopy parameter is required. No such result is known to us, where such an "equicontinuity" is required only on the fixed point set: Since in our application the set Ω is unbounded, it is crucial for us that we do not require equicontinuity of the whole family

$$\big\{h(\cdot, x):\ x \in \overline{\Omega}\,\big\}$$

(as was done by the global Lipschitz assumption in [5]) but only for the subfamily (2.3). So, although the proof is rather straightforward and along the lines of [2,5], we will give some details.

Theorem 1 *(Continuation theorem for contractions). Let X be a metric space, and let $\Omega \subseteq X$ be an open subset. Let I be a metric space and $h: I \times \overline{\Omega} \to X$ be such that*

$$M := \overline{h(I \times \Omega)} \cap \overline{\Omega}$$

is complete. Let

$$\operatorname{Fix}(h) := \bigl\{x \in \Omega\colon\ x \in h\bigl(I \times \{x\}\bigr)\bigr\}$$

denote the joint fixed point set of h. Suppose that $h(\cdot, x)$ is continuous on I for each $x \in \Omega \cap \overline{\operatorname{Fix}(h)}$ and that

$$\bigl\{h(\cdot, x)\colon\ x \in \Omega \cap \operatorname{Fix}(h)\bigr\} \tag{2.3}$$

is even equicontinuous on I. Suppose also that there is some $q \in [0, 1)$ such that

$$d\bigl(h(t, x), h(t, y)\bigr) \leqslant q d(x, y) \quad (x, y \in \overline{\Omega},\ t \in I), \tag{2.4}$$

and that there is some $t_0 \in I$ such that $h(t_0, \cdot)$ maps a nonempty closed subset $C \subseteq \overline{\Omega}$ into itself.

Let $J \subseteq I$ be a connected subset of I satisfying $t_0 \in J$, and let $\overline{J}$ denote its closure in I. Finally, assume that $h(t, x) \neq x$ for all $t \in J$ and all $x \in \partial\Omega$.

Then $h(t, \cdot)$ has exactly one fixed point $x(t)$ for each $t \in \overline{J}$ which depends continuously on t. More precisely,

$$d\bigl(x(t), x(s)\bigr) \leqslant \frac{d(h(t, x(s)), h(s, x(s)))}{1 - q} \quad (t, s \in \overline{J}). \tag{2.5}$$

Since we do not require that I is compact, we should emphasize that by equicontinuity of a family F of functions in metric spaces we mean that for each $\varepsilon > 0$ there is some $\delta > 0$ such that $d(f(t), f(s)) \leqslant \varepsilon$ whenever $f \in F$ and $d(t, s) \leqslant \delta$, i.e. δ is not only independent of f but *simultaneously* independent of t and s (in particular, each $f \in F$ is uniformly continuous).

Proof. The uniqueness of the fixed point of $h(t, \cdot)$ is a straightforward consequence of (2.4). Suppose that the first alternative is false. Let I_* be the set of all $t \in I$ for which a (unique) fixed point $x(t) \in \overline{\Omega}$ of $h(t, \cdot)$ exists, i.e. $x(t) \in M$, and for $t \in J$ by hypothesis also $x(t) \in \Omega$.

$h(t_0, C) \subseteq C$ implies in view of the continuity of $h(t_0, \cdot)$ that $C \subseteq M$, and so C is complete. Banach's fixed point theorem thus implies $t_0 \in I_*$.

Let now $t \in I_* \cap J$. Since Ω is open, there is some $r > 0$ such that $\overline{\Omega}$ contains the closed ball

$$B_t := \bigl\{y \in X\colon\ d\bigl(y, x(t)\bigr) \leqslant r\bigr\}.$$

By the equicontinuity of (2.3), we find some neighborhood $U_t \subseteq I$ of t such that for all $s \in U_t$ the estimate

$$d\bigl(h\bigl(s, x(t)\bigr), h\bigl(t, x(t)\bigr)\bigr) \leqslant (1 - q)r$$

holds. For all $s \in U_t$ and all $y \in B_t$, we conclude

$$\begin{aligned} d\bigl(h(s, y), x(t)\bigr) &\leqslant d\bigl(h(s, y), h\bigl(s, x(t)\bigr)\bigr) + d\bigl(h\bigl(s, x(t)\bigr), h\bigl(t, x(t)\bigr)\bigr) \\ &\leqslant q d\bigl(y, x(t)\bigr) + (1 - q)r \leqslant r, \end{aligned}$$

i.e. $h(s, \cdot)$ maps B_t into itself. In particular, $B_t \subseteq M$ is complete, and so $h(s, \cdot)$ has a fixed point by Banach's fixed point theorem. Hence, $U_t \subseteq I_*$. In particular, $I_* \cap J$ is open in J.

The estimate (2.5) holds for all $t, s \in I_*$ by

$$\begin{aligned} d\big(x(t),x(s)\big) &\leqslant d\big(h\big(t,x(t)\big),h\big(t,x(s)\big)\big)+d\big(h\big(t,x(s)\big),x(s)\big) \\ &\leqslant qd\big(x(t),x(s)\big)+d\big(h\big(t,x(s)\big),h\big(s,x(s)\big)\big). \end{aligned}$$

This implies in view of the uniform equicontinuity of (2.3) in particular that $x : I_* \to M$ is uniformly continuous. Since M is complete, x has an (unique) extension to a continuous function $x : \bar{I}_* \to M$. Moreover, if $t_n \in I_*$ converges to some $t \in \bar{I}_*$, then

$$\begin{aligned} d\big(x(t),h\big(t,x(t)\big)\big) &\leqslant d\big(x(t),x(t_n)\big)+d\big(h\big(t_n,x(t_n)\big),h\big(t_n,x(t)\big)\big) \\ &\quad + d\big(h\big(t_n,x(t)\big),h\big(t,x(t)\big)\big) \\ &\leqslant d\big(x(t),x(t_n)\big)+qd\big(x(t_n),x(t)\big)+d\big(h\big(t_n,x(t)\big),h\big(t,x(t)\big)\big) \to 0 \end{aligned}$$

as $n \to \infty$, i.e. $x(t)$ is a fixed point of $h(t,\cdot)$. In particular, $t \in I_*$. Hence, I_* is closed in I. Since J is connected and $I_* \cap J$ is nonempty (it contains t_0), closed, and open in J, we conclude $I_* \supseteq J$. Since I_* is closed in I, we conclude even $I_* \supseteq \bar{J}$. □

When $I = [0,1]$ and h is a convex homotopy between two operators T_0 and T_1, one obtains the Leray–Schauder alternative. Note that "classically" $T_0(x) \equiv x_0$ is a constant map, but in our application, T_0 will be a map associated to problem (1.1)/(1.2) in the trivial case $u_0 = 0$.

We point out once more that it is important for our application that we do not require any boundedness for Ω or $T_i(\Omega)$ but only an a priori bound for an auxiliary fixed point set.

For the case $T_0(x) \equiv 0$ and bounded Ω (respectively a ball Ω), the following Corollary 1 is contained in [5] and [2], respectively. A variant for unbounded Ω (and $T_0(x) \equiv 0$) was considered in [4]. Note that for bounded Ω the boundedness of (2.7) is no additional requirement, since T_0 and T_1 are bounded maps.

Corollary 1 *(Leray–Schauder for contractions). Let X be a normed space, $\Omega \subseteq X$ open and nonempty and such that $\overline{\Omega}$ is complete. Let $T_0, T_1 : \overline{\Omega} \to X$ be two contractions such that the following Leray–Schauder type boundary condition holds*:

$$T_1(x) - T_0(x) \neq \lambda\big(x - T_0(x)\big) \quad (\lambda > 1,\ x \in \partial\Omega). \tag{2.6}$$

Assume in addition that the set

$$\bigcup_{\lambda>1}\big\{T_1(x) - T_0(x)\colon\ x \in \Omega \text{ and } T_1(x) - T_0(x) = \lambda\big(x - T_0(x)\big)\big\} \tag{2.7}$$

is bounded. If T_0 has a fixed point in Ω then T_1 has a fixed point in $\overline{\Omega}$, and moreover, each of the maps

$$T_t(x) := tT_1(x) + (1-t)T_0(x) \quad (0 \leqslant t \leqslant 1) \tag{2.8}$$

has a unique fixed point $x(t) \in \overline{\Omega}$ which depends Lipschitz continuously on $t \in [0,1]$.

Proof. Put $I := [0,1]$, $h(t,x) := T_t(x) = T_0(x) + t(T_1(x) - T_0(x))$, $t_0 := 0$, and $J := [0,1)$ in Theorem 1. Note that, since T_i has at most one fixed point for $i = 0, 1$, the boundedness of (2.7) is equivalent to the boundedness of the (by at most two elements larger) set

$$\bigcup_{t\in I}\big\{T_1(x) - T_0(x)\colon\ x \in \Omega \text{ and } t\big(T_1(x) - T_0(x)\big) = x - T_0(x)\big\}. \tag{2.9}$$

Hence, the boundedness of (2.7) is equivalent to the uniform equicontinuity of (2.3); the family (2.3) is even Lipschitz continuous with a uniform Lipschitz constant. Hypothesis (2.6) means that $h(t,\cdot)$ has for $t \in (0,1)$ no fixed point on $\partial\Omega$. Since $h(0,\cdot) = T_0$ has a fixed point in Ω, the uniqueness of the fixed point implies that $h(0,\cdot)$ has no fixed point on $\partial\Omega$ either. Hence, Theorem 1 applies. The Lipschitz continuity of $x(t)$ with respect to $t \in \overline{J} = [0,1]$ follows from the subsequent formula (2.10). $\square$

Note that if T_0 and T_1 are both contractions with contraction constant q_0 and q_1, respectively, then the map h in the above proof satisfies (2.4) with $q := \max\{q_0, q_1\}$, and (2.5) simply reads

$$\big\|x(t) - x(s)\big\| \leqslant \frac{\|T_1(x(s)) - T_0(x(s))\|}{1-q}|t-s| \leqslant \frac{M}{1-q}|t-s| \quad \big(t, s \in [0,1]\big), \tag{2.10}$$

where M is an upper bound for the norm of the elements of (2.9).

The suggested algorithm to obtain the fixed point is now as follows. One has to start with successive approximations with small $t > 0$ for T_t (beginning e.g. with some value sufficiently close to the unique fixed point $x(0)$ of T_0) to obtain an approximation for $x(t)$, increase t a bit, doing some successive approximations again with T_t (starting with the previous approximation), increase t, and so on, until $t = 1$ is reached after a finite number of steps. Then successive approximations can be repeated for T_1 to improve the final approximation. Our proof can easily be modified to show that this algorithm converges to the fixed point of T_1 if

(1) in each step sufficiently many successive approximations are done so that the current approximations are sufficiently close to the fixed point $x(t)$, and
(2) in each step, t is not increased too much (so that our current approximation belongs to the invariant ball B_t considered in the proof of Theorem 1).

To verify the first of these conditions, one can use the known a priori bound for Banach's fixed point theorem. To find an estimate on how much t can be increased, one has to take the estimate (2.10) into account.

3. The linear problem

In this section, we summarize some observations about the linear boundary value problem

$$\Delta_r u(r) = y(r) + c^2 u(r) \tag{3.1}$$

on $[R,\infty)$ with boundary values (1.2) which might e.g. be considered as a linearization of (1.1) when $c^2 = f'(0) > 0$, or at least "close" to a linearization if $-f' \geqslant 0$ is "small" near 0.

We assume throughout that $c > 0$. For this case, the corresponding Green's function for (3.1) was calculated in [1]. To formulate this result, we fix throughout a continuous weight function $w : [R,\infty) \to (0,\infty)$ with

$$\lim_{r\to\infty} w(r) = \infty \tag{3.2}$$

and work in the Banach space $C_w([R,\infty))$ of all continuous functions $x : [R,\infty) \to \mathbb{R}$ with the corresponding weighted norm

$$\|x\|_w := \sup_{r\in[R,\infty)} \big|w(r)x(r)\big|. \tag{3.3}$$

Let I_α and K_α denote the modified Bessel functions of the first and second kind of order α, respectively. In particular, I_α and K_α are positive on $(0,\infty)$, and K_α decays exponentially fast near ∞. It turns out that, under reasonable mild assumptions, the solution of problem (3.1) is given by the affine operator

$$\begin{aligned}A_{u_0,c,R}y(r) :=& -\left(\int_r^\infty K_0(ct)y(t)t\,dt\right)I_0(cr) - \left(\int_R^r I_0(ct)y(t)t\,dt\right)K_0(cr)\\ &+ \frac{1}{K_0(cR)}\left(u_0 + I_0(cR)\left(\int_R^\infty K_0(ct)y(t)t\,dt\right)\right)K_0(cr).\end{aligned}$$

More precisely, assume in addition to (3.2) that w is such that

$$\limsup_{r\to\infty} K_0(cr)w(r) < \infty \tag{3.4}$$

and that the quantity

$$\begin{aligned}L_w(c,R) := \sup_{r\in[R,\infty)}\Biggl(&\left(\int_r^\infty K_0(ct)\frac{t}{w(t)}\,dt\right)I_0(cr)w(r)\\ &+\left(\int_R^r I_0(ct)\frac{t}{w(t)}\,dt\right)K_0(cr)w(r)\\ &+\frac{I_0(cR)}{K_0(cR)}\left(\int_R^\infty K_0(ct)\frac{t}{w(t)}\,dt\right)K_0(cr)w(r)\Biggr)\end{aligned}$$

is finite. The asymptotic behaviour of the Bessel functions implies that $w(r) := r$ always has this property. The following result was obtained in [1].

Theorem 2. *Suppose that* (3.2) *and* (3.4) *hold and that* $L_w(c,R) < \infty$. *Then the affine operator* $A_{u_0,c,R}$ *maps* $C_w([0,\infty))$ *into itself, and the norm of its linear part is bounded by* $L_w(c,R)$, *i.e.*

$$\left\| A_{u_0,c,R}(x) - A_{u_0,c,R}(y)\right\|_w \leqslant L_w(c,R)\|x-y\|_w.$$

Moreover, for each $y \in C_w([0,\infty))$, *the function* $x := A_{u_0,c,R}(y)$ *is the only solution of the linear problem* (3.1)/(1.2).

As mentioned above, the asymptotic behaviour of the Bessel functions I_0 and K_0 implies that the function $w(r) := r$ is a good choice. More precisely, the following result was shown in [1].

Proposition 1. *For* $w := \mathrm{id}$, *there is a universal finite smallest constant* $L > 1$ *such that, for all* $c, R > 0$,

$$L_{\mathrm{id}}(c,R) = \frac{1}{c^2}L_{\mathrm{id}}(1,cR) \leqslant \frac{L}{c^2}.$$

The calculations in [1] suggest that one indeed obtains the best possible upper estimates for $L_w(c,R)$ when the growth of w is asymptotically linear. Moreover, by numerical experiments, it seems that the constant in Proposition 1 satisfies $L \leqslant 1.2$; actually, 1.2 even seems to be a rather generous upper bound for L.

In this paper, the quantity

$$\widetilde{L}_w(c,R) := \sup_{\widetilde{R}\in[R,\infty)} L_w(c,\widetilde{R})$$

will play a crucial role. The definition and Proposition 1 imply immediately that, with $w = \mathrm{id}$ and the same constant L as above,

$$L_{\mathrm{id}}(c,R) \leqslant \widetilde{L}_{\mathrm{id}}(c,R) = \frac{1}{c^2}\widetilde{L}_{\mathrm{id}}(1,cR) \leqslant \frac{L}{c^2} \tag{3.5}$$

holds for all $c, R > 0$.

4. The nonlinear problem

Throughout this section, let w, c, and R be as in the previous section such that (3.2) and (3.4) hold and that $\widetilde{L}_w(c,R) < \infty$.

Let $I_f \subseteq \mathbb{R}$ be some closed (but not necessarily bounded) interval with interior $\mathring{I}_f$ such that $0 \in \mathring{I}_f$, and let $f : I_f \to \mathbb{R}$ satisfy the following properties with some constants $c > 0$ and $L_f \in [0, 1/\widetilde{L}_w(c,R))$:

(1) $f(0) = 0$.
(2) $f_c(u) := f(u) - c^2u$ is Lipschitz in I_f with constant at most L_f, i.e.

$$\left|f_c(u) - f_c(v)\right| \leqslant L_f|u - v| \quad (u, v \in I_f). \tag{4.1}$$

(3) f_c is nonnegative on the positive part of I_f:

$$f_c(u) \geqslant 0 \quad (0 < u < \sup I_f). \tag{4.2}$$

Note that (4.1) implies in particular that f is a Lipschitz function on I_f. This condition is satisfied by the intermediate value theorem if f_c is continuous on I_f and differentiable in $\mathring{I}_f$ with

$$\left|f'(u) - c^2\right| \leqslant L_f \quad (u \in \mathring{I}_f).$$

In particular, if $f : \mathbb{R} \to \mathbb{R}$ is C^1 with $f(0) = 0$ and $f'(0) > 0$, one can for each $L_f > 0$ find a number $c > 0$ (e.g. slightly smaller than $\sqrt{f'(0)}$) and a corresponding interval $I_f = (-\varepsilon, \varepsilon)$ such that (4.1) and (4.2) hold.

In particular, in view of (3.5), for each C^1 function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and $f'(0) > 0$ the above hypotheses can be satisfied on some interval I_f with $w := \mathrm{id}$ and $c \approx \sqrt{f'(0)}$.

In view of Corollary 1, we put $X := C_w([R,\infty))$ and define

$$\Omega := \left\{x \in C_w\big([R,\infty)\big) \colon x\big([R,\infty)\big) \subseteq \mathring{I}_f\right\}. \tag{4.3}$$

Note that Ω is open in the Banach space X, but for each non-degenerate interval I_f unbounded (even if I_f is bounded) so that "classical" continuation theorems do usually not apply for set-contractions on Ω. However, Corollary 1 will. The following result is a straightforward consequence of (4.1).

Lemma 1 *(Contraction). The superposition operator $F_c(u)(r) := f_c(u(r))$ sends $\overline{\Omega}$ into $C_w([R,\infty))$ and is Lipschitz with constant at most L_f. In particular, for any $u_0 \in \mathbb{R}$, the operator*

$$T_{u_0,c,R} := A_{u_0,c,R} \circ F_c : \overline{\Omega} \to C_w\big([R,\infty)\big) \tag{4.4}$$

is a contraction with constant at most $q := L_w(c,R)L_f \leqslant \widetilde{L}_w(c,R)L_f$.

The following result is similarly to the uniqueness result in [1], although our hypotheses are slightly different.

Lemma 2 *(Uniqueness). For each $\widetilde{R} \geqslant R$ and each $\tilde{u}_0 \in I_f$ the boundary value problem* (1.1) *on $[\widetilde{R}, \infty)$ with boundary values*

$$u(\widetilde{R}) = \tilde{u}_0, \qquad \lim_{r\to\infty} u(r) = 0 \tag{4.5}$$

has at most one solution in the space $C_w([\widetilde{R}, \infty))$ such that $u([\widetilde{R}, \infty)) \subseteq I_f$.

Proof. Since $\widetilde{L}_w(c, R) \geqslant \widetilde{L}_w(c, \widetilde{R})$, we have $L_f \in [0, 1/\widetilde{L}_w(c, \widetilde{R})]$. Hence, all our hypotheses are satisfied when we replace R by $\widetilde{R}$. Thus, without loss of generality, we may assume $\widetilde{R} = R$ and $\tilde{u}_0 = u_0$.

Let u be a solution of the corresponding problem (1.1)/(1.2) such that $u \in C_w([R, \infty))$ and $u([\widetilde{R}, \infty)) \subseteq I_f$. Then $u \in \overline{\Omega}$, and thus Lemma 1 implies that $y(r) := f_c(u(r))$ belongs to $C_w([\widetilde{R}, \infty))$. Moreover, Theorem 2 implies $u = A_{u_0,c,R}\, y = (A_{u_0,c,R} \circ F_c)u$. Thus, each such solution u is a fixed point of the contraction (4.4). It remains to observe that contractions have at most one fixed point. □

We emphasize that all results obtained so far hold also if (4.2) fails. However, we use (4.2) for the following a priori estimates.

Lemma 3 *(A priori estimates). A function u is a solution of problem* (1.1)/(1.2) *in the space $C_w([R, \infty))$ and satisfies $u([\widetilde{R}, \infty)) \subseteq I_f$ if and only if u is a fixed point of* (4.4). *In this case, the following holds if the boundary value u_0 is nonnegative*:

(1) *If $u_0 = 0$, then $u(r) \equiv 0$.*
(2) *If $u_0 > 0$, then*

$$0 < u(r) < u_0 \quad (R < r < \infty) \tag{4.6}$$

and

$$0 < u(r) \leqslant \frac{u_0 + B_{c,R} B_{f_c,u_0}}{K_0(cR)} K_0(cr) \quad (R \leqslant r < \infty), \tag{4.7}$$

where

$$B_{c,R} := I_0(cR) \int_R^\infty K_0(ct)t\,dt$$

and

$$B_{f_c,u_0} := \max_{u\in[0,u_0]} f_c(u).$$

In addition, the right inequality of (4.7) *is strict if $B_{f_c,u_0} > 0$.*

Proof. We have seen in the proof of Lemma 2 that all such solutions u as above are fixed points of (4.4). Conversely, if $u \in \overline{\Omega}$ is such a fixed point, then we have $u = A_{u_0,c,R}(y)$ with $y := F_c(u)$, and so Theorem 2 implies that u solves (1.1)/(1.2).

Assume now $u_0 \geqslant 0$. If there is some $\widetilde{R} \geqslant R$ with $u(\widetilde{R}) = 0$, then there is some $\widetilde{R} \geqslant R$ with $u(\widetilde{R}) = 0$. In particular, the restriction of u to $[\widetilde{R}, \infty)$ satisfies the boundary value problem (1.1)/(4.5) with $\tilde{u}_0 = 0$. Since this problem has the trivial solution $\tilde{u}(r) \equiv 0$, Lemma 2 implies $u = \tilde{u} = 0$ on $[\widetilde{R}, \infty)$. In particular, fixing some $R_0 > \widetilde{R}$, u satisfies the second order initial value problem (1.1) with initial values $u(R_0) = 0$, $u'(R_0) = 0$. Since f satisfies a Lipschitz condition, this problem has at most one solution on $[R, \infty)$. Since the trivial solution solves this problem, we conclude $u \equiv 0$ on $[R, \infty)$ which is possible if and only if $u_0 = 0$. This shows that $u(r) \equiv 0$ for $u_0 = 0$ as well as the left inequality in (4.6).

Since u is nonnegative on $[R, \infty)$, hypothesis (4.2) implies that $y := F_c(u)$ is nonnegative on $[R, \infty)$. Since u is a fixed point of (4.4), the definition of $A_{u_0,c,R}$ shows that, for all $r \in [R, \infty)$,

$$u(r) = -\left(\int_r^\infty I_0(cr)K_0(ct)y(t)t\,dt\right) - \left(\int_R^r I_0(ct)K_0(cr)y(t)t\,dt\right) + \frac{K_0(cr)}{K_0(cR)}\left(u_0 + \left(\int_R^\infty I_0(cR)K_0(ct)y(t)t\,dt\right)\right). \tag{4.8}$$

Recall that $I_0' = I_1$ and $K_0' = -K_1$, see e.g. [12, Section 3.7.1(7)]. In particular, I_0 is strictly increasing and K_0 is strictly decreasing on $(0, \infty)$ (and both functions are positive on $(0, \infty)$). This implies on the one hand (the estimate being strict in view of $u_0 > 0$)

$$\begin{aligned} u(r) &< -\left(\int_r^\infty I_0(cR)K_0(ct)y(t)t\,dt\right) - \left(\int_R^r I_0(cR)K_0(ct)y(t)t\,dt\right) \\ &\quad + 1 \cdot \left(u_0 + \left(\int_R^\infty I_0(cR)K_0(ct)y(t)t\,dt\right)\right) \\ &= u_0, \end{aligned}$$

i.e. the right inequality of (4.6) holds. On the other hand, since we have just shown $u([R, \infty)) \subseteq [0, u_0]$, we obtain $y([R, \infty)) \subseteq [0, B_{f_c,u_0}]$, and hence (4.8) implies, since I_0, and K_0 are nonnegative,

$$u(r) \leqslant -0 - 0 + \frac{K_0(cr)}{K_0(cR)}\left(u_0 + I_0(cR)\int_R^\infty K_0(ct)B_{f_c,u_0}t\,dt\right).$$

The estimate for the last integral is strict if $B_{f_c,u_0} > 0$, because $y \in C_w([R, \infty))$ implies in particular $y(t) \to 0$ as $t \to \infty$ and thus $y(t) < B_{f_c,u_0}$ on a set of positive measure. □

Now we are in a position to formulate our main result:

Theorem 3. *Let $c, R > 0$, and let $w : [R, \infty) \to (0, \infty)$ be continuous such that* (3.2) *and* (3.4) *hold and $\widetilde{L}_w(c, R) < \infty$. Let I_f be a closed interval with $0 \in \mathring{I}_f$, and let $f : I_f \to \mathbb{R}$ satisfy $f(0) = 0$ and such that $f_c(u) := f(u) - c^2u$ satisfies* (4.1) *with some $L_f \in [0, 1/\widetilde{L}_w(c, R))$ and* (4.2).

Then problem (1.1)/(1.2) *has for each boundary value $u_0 \in I_f \cap [0, \infty)$ exactly one solution u in $C_w([R, \infty))$ with $u([R, \infty)) \subseteq I_f$.*

In case $u_0 = 0$ this is the trivial solution, and in case $u_0 > 0$, this solution satisfies $u((R,\infty)) \subseteq (0,u_0)$ and (4.7) (*the inequality being strict if $B_{f_c,u} > 0$, i.e. if $f(u) \neq c^2u$ for some $u \in [0,u_0]$*).

The solution depends with respect to the norm (3.3) *locally Lipschitz continuous on the boundary value $u_0 \in I_f \cap [0,\infty)$.*

Proof. We show that we can apply Corollary 1 in the space $X := C_w([R,\infty))$ with Ω as in (4.3) and the operators $T_t := T_{tu_0,c,R} = A_{tu_0,c,R} \circ F_c$. Observe that (2.8) holds, since only the inhomogeneous part of $A_{u_0,c,R}$ depends on u_0, and this dependence is linear. Note that, by our previous results, T_1 corresponds to problem (1.1)/(1.2), and T_0 corresponds to the same problem with the trivial boundary value $u_0 = 0$. More precisely, the fixed points of T_t are precisely the solutions $u \in \overline{\Omega}$ of problem (1.1) with boundary values

$$u(R) = tu_0, \qquad \lim_{r\to\infty} u(r) = 0.$$

Lemma 3 implies in particular $u([R,\infty)) \subseteq [0,tu_0]$. Since for $t \in [0,1)$ we have $[0,tu_0] \subseteq \mathring{I}_f$, we conclude that $u \in \Omega$, i.e. for $t \in [0,1)$ the operator T_r has no fixed point u on $\partial\Omega$. Moreover, for the norm of all fixed points of T_t $(0 \leqslant t \leqslant 1)$, we have an a priori estimate by (4.7). Indeed, for all $t \in [0,1]$ and all fixed points u of T_t, we have by (4.8) the estimate

$$0 \leqslant u(r) \leqslant \frac{tu_0 + B_{c,R}B_{f_c,tu_0}}{K_0(cR)} K_0(cr) \leqslant \frac{u_0 + B_{c,R}B_{f_c,u_0}}{K_0(cR)} K_0(cr) \quad (R \leqslant r < \infty).$$

Since the function $h(r) := K_0(cr)$ belongs to X by (3.4), we obtain the uniform norm estimate

$$\|u\|_w \leqslant \frac{u_0 + B_{c,R}B_{f_c,u_0}}{K_0(cR)} \|h\|_w$$

for all those fixed points. Finally, T_0 has the trivial fixed point which in view of $0 \in \mathring{I}_f$ belongs to Ω. Hence, all hypotheses of Corollary 1 are satisfied. □

Our proof shows that an upper estimate for the Lipschitz constant in Theorem 3 with respect to boundary value u_0 on an interval $[0,\tilde{u}_0]$ is given by

$$\frac{\tilde{u}_0 + B_{c,R}B_{f_c,\tilde{u}_0}}{K_0(cR)(1 - L_w(c,R)L_f)} \sup_{r\in[R,\infty)} \bigl(w(r)K_0(cr)\bigr).$$

We point out that, for our above choice of T_t, the algorithm suggested after Corollary 1 means that one should first approximate the solution of (1.1)/(1.2) for a *small* boundary value u_0 (using successive approximations, starting with the trivial solution), then increase u_0 a bit, do successive approximations, and so on, until one reaches the required boundary value u_0 for which, finally, the method of successive approximations will converge to the solution.

5. The special case (1.3)

We discuss now the hypotheses of Theorem 3 for the special case of the function (1.3). Of course, the aim is to choose the interval I_f as large as possible such that all other hypotheses of Theorem 3 are satisfied with an appropriate choice of c and w, i.e. such that we can conclude that problem (1.1)/(1.2) has a solution for each boundary value $u_0 \in I_f \cap [0,\infty)$. Some of the following calculations are similar to [1], but we have to be more careful now, since we must also take care that the sign condition (4.2) is satisfied which was not needed for the uniqueness result in [1].

It is necessary for (4.2) (for any choice of I_f with $0 \in \mathring{I}_f$) that

$$f'(0) \geqslant c^2. \tag{5.1}$$

Conversely, if (5.1) holds, then $f_c'(u) = a\alpha\cosh(u) - c^2 \geqslant a\alpha - c^2 = f'(0) - c^2 \geqslant 0$ implies that f_c is nondecreasing and thus nonnegative on $[0, \infty)$. Consequently, condition (4.2) is equivalent to (5.1), independent of the particular choice of I_f.

In the following considerations, we will restrict ourselves to the case $w(u) := u$, and concerning the quantity $\widetilde{L}_w(c, R)$, we will only use the estimate (3.5) where $L > 1$ is the universal constant of Proposition 1. This means that we replace the hypothesis $L_f \in [0, 1/\widetilde{L}_w(c, R))$ by the slightly more restrictive hypothesis

$$\lambda := \frac{c^2}{L_f} \in (L, \infty). \tag{5.2}$$

Since f is a C^1 function, hypothesis (4.1) holds on the closed interval I_f with the constants $c > 0$ and $L_f = \lambda c^2$ if (and only if)

$$\left|f_c'(u)\right| \leqslant L_f = \frac{c^2}{\lambda} \quad (u \in I_f).$$

Using the definition of f_c and f, this can equivalently be rewritten as

$$(\lambda - 1)c^2 \leqslant \lambda a\alpha\cosh(au) \leqslant (\lambda + 1)c^2 \quad (u \in I_f). \tag{5.3}$$

Note that (5.1) implies $a\alpha\cosh(au) \geqslant f'(0) \geqslant c^2$, and so the left inequality of (5.3) is satisfied automatically if (5.1) holds. Since $0 \in \mathring{I}_f$, a necessary condition for our choice of the constants must be $\lambda a\alpha\cosh(0) \leqslant (\lambda + 1)c^2$, i.e.

$$f'(0) \leqslant \left(1 + \lambda^{-1}\right)c^2. \tag{5.4}$$

Suppose in the following that (5.4) holds. Since $\cosh : [0, \infty) \to [1, \infty)$ is a bijection, we conclude that the right inequality of (5.3) is equivalent to

$$|u| \leqslant \frac{1}{a}\cosh^{-1}\frac{(1 + \lambda^{-1})c^2}{a\alpha} \quad (u \in I_f),$$

where $\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$ denotes the inverse function to cosh. Thus, concerning hypothesis (4.1), the maximal choice for the interval I_f is

$$I_f := \left[-m(\lambda, c), m(\lambda, c)\right]$$

where

$$m(\lambda, c) := \frac{1}{a}\log\frac{(1 + \lambda^{-1})c^2 + \sqrt{(1 + \lambda^{-1})^2 c^4 - f'(0)^2}}{f'(0)}.$$

Our aim is to choose c and λ (and thus implicitly L_f) such that all above hypotheses are satisfied and such that the interval I_f (i.e. $m(\lambda, c)$) becomes as large as possible. This is certainly the case if we choose c as large as possible, i.e. if $c^2 = f'(0)$ by (5.1). Note that for this choice (5.4) is automatically satisfied. Thus, the only restriction concerning λ is $\lambda > L$. We conclude that, with an appropriate choice of λ, we can get as close to the value $m(L, \sqrt{f'(0)})$ as we want. Putting

$$C := \log\left(\left(1 + L^{-1}\right) + \sqrt{\left(1 + L^{-1}\right)^2 - 1}\right), \tag{5.5}$$

we thus can apply Theorem 3 whenever $I_f \subseteq (-C/a, C/a)$. Hence, the following result follows for $u_0 \in [0, C/a)$; for symmetry reasons, an analogous conclusion holds for $u_0 \in (-C/a, 0]$ which we also include in the following formulation.

Theorem 4. *Let $L \approx 1.2$ be the universal constant of Theorem 2, and let $C \approx 1.21$ be given by (5.5). Let f be given by (1.3) with constants $a, \alpha > 0$.*

Then for each boundary value u_0 with $|u_0| < C/a$, problem (1.1)/(1.2) has exactly one solution u satisfying $u(r) = O(1/r)$ as $r \to \infty$ and $u([R, \infty)) \subseteq (-C/a, C/a)$.

The solution actually satisfies $|u(r)| = O(K_0(r\sqrt{a\alpha}))$ as $r \to \infty$, and $\operatorname{sgn} u(r) = \operatorname{sgn} u_0$ and $|u(r)| \leqslant |u_0|$ for all $r \in [R, \infty)$, the inequality being strict for $r > R$ and $u_0 \neq 0$. The solution can be approximated by the algorithm sketched after Theorem 3.

The estimate $|u(r)| = O(K_0(r\sqrt{a\alpha}))$ stems from (4.7); the constant B_{f_c,u_0} becomes in our case, i.e. for f as in (1.3) and $c^2 = f'(0) = a\alpha$,

$$B_{f_c,u_0} = \max_{u \in [0,|u_0|]} \alpha\bigl(\sinh(au) - au\bigr) = \alpha\bigl(\sinh\bigl(a|u_0|\bigr) - a|u_0|\bigr).$$

Let us briefly compare Theorem 4 with the existence and the uniqueness result of [1].

Our hypotheses $|u_0| < C/a$ is a bit more restrictive than the hypothesis for the uniqueness result in [1]: By that result the uniqueness statement holds even if one replaces C by

$$\widetilde{C} := \log \frac{\sqrt{L} + 1}{\sqrt{L} - 1} \approx 3.$$

As explained above, this is due to the fact that for our proof of the a priori estimates we needed the sign condition (4.2) which is actually not needed for the proof of the uniqueness statement.

However, the corresponding existence result in [1] has the hypothesis $|u_0| \leqslant C_{R,a,\alpha}/a$ where

$$C_{R,a,\alpha} := \sup_{\lambda \in (L,\infty)} R\left(R + \frac{L}{\lambda - L} \sup_{r \in [R,\infty)} \frac{K_0(c_{\lambda,a,\alpha} r) r}{K_0(c_{\lambda,a,\alpha} R)}\right)^{-1} \log \frac{\sqrt{\lambda} + 1}{\sqrt{\lambda} + 1}$$

and $c_{\lambda,a,\alpha} := \sqrt{\lambda a\alpha/(\lambda - 1)}$. Estimating the inner supremum generously from below by R, we thus find that a necessary condition for this hypothesis is $|u_0| \leqslant C_0/a$ where

$$C_0 := \sup_{\lambda \in (L,\infty)} \frac{\lambda - L}{\lambda} \log \frac{\sqrt{\lambda} + 1}{\sqrt{\lambda} + 1} \approx 0.8.$$

Hence, even this (very generous) necessary condition for the existence result in [1] is more restrictive than our hypothesis of Theorem 4, i.e. the existence statement in Theorem 4 is strictly stronger than that of [1] (for each choice of a, α, and R).

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